POLARIZATION AND RIEMANN SURFACES

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Abstract

Under conditions, such that geometrical optics can be applied, anisotropical effects in a cold magnetoplasma are described by the so–called magneto–ionic theory in terms of a refractive index and polarization, both complex and double valued.

This ambiguity, resulting from the birefractivity of the plasma, can be eliminated by conformal mapping onto a double sheet Riemann surface. The mapping is evaluated and a physical interpretation of branch points and the branch cut is given. When crossing the branch cut the wave nomenclature must be changed.

1 Introduction

Before entering into the theory of complex and double valued polarization, some introductory remarks are necessary.

Firstly, throughout this text a 'wave–fixed' orthogonal system of coordinates will be used. Thus the wave normal $\vec{n}$ is one of the basis vectors of our system, corresponding to the z–axis. The other two, x and y, may be rotated around the z–axis so that we are free to impose a further condition. The axes are, therefore, chosen such that the magnetic field vector ($\vec{B}$) is in the y–z plane at an angle θ with the z–axis.
Secondly, both refractive index and polarization can be expressed by means of the 'dielectric eigenvalues'.
The dielectric tensor is a tensor of the form

$$\hat{\varepsilon} = \varepsilon_+ \mathcal{U} + (\varepsilon_0 - \varepsilon_+)\vec{e}\,\vec{e} + i\varepsilon_- \vec{e} \times \mathcal{U}\,,$$

with $\varepsilon_\pm = \frac{1}{2}(\varepsilon_{+1} \pm \varepsilon_{-1})$; $\mathcal{U}$...unit tensor; $\vec{e}$...unit vector; $i = \sqrt{-1}$ where ε_0,$\varepsilon_{\pm 1}$ are the eigenvalues of $\hat{\varepsilon}$. In the following, a fairly widely used notation introduced by Stix [1962] will be used

$$\varepsilon_0 \equiv P,\ \varepsilon_{+1} \equiv R,\ \varepsilon_{-1} \equiv L\,,$$
$$\varepsilon_+ \equiv S,\ \varepsilon_- \equiv D\,.$$

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The point of departure of the following discussion on polarization is the so–called polarization equation of magneto–ionic theory for the transverse component of the electric field vector ($\vec{E}$)

$$\left(\frac{E_y}{E_x}\right)^2 + i\frac{\sin^2\theta}{\cos\theta}\frac{RL-PS}{PD}\left(\frac{E_y}{E_x}\right) + 1 = 0\,. \tag{1.1}$$

Using abbreviations

$$Q \equiv \frac{E_x}{E_y} \qquad \text{and} \qquad \zeta \equiv \frac{2\cos\theta}{\sin^2\theta}\frac{PD}{RL-PS}\,, \tag{1.2}$$

the above equation can be given as

$$\frac{1}{Q^2} - \frac{2}{i\zeta Q} + 1 = 0\,.$$

This is a quadratic equation for 1/Q showing the well known fact that, in a cold magnetoplasma, a progressive wave must have a polarization given by one of the two solutions

$$Q = \frac{i\zeta}{1 \pm \sqrt{1+\zeta^2}} = \frac{1 \mp \sqrt{1+\zeta^2}}{i\zeta}\,. \tag{1.3}$$

Hence, the birefractivity of the medium gives rise to a sort of ambiguity as there are two Q–values for each ζ–value, i.e. two characteristic polarizations. This ambiguity, however, can be eliminated by introducing a double sheet Riemann surface for ζ.

2 The double sheet Riemann surface

These two sheets, corresponding to the two branches of $Q(\zeta)$, touch each other at the so–called branch points; points where the two Q–values are identical, i.e. where both characteristic polarizations are the same.
Any curve connecting the branch points may be adopted as branch cut, and along this cut the lower edge of the bottom sheet is attached to the upper edge of the top sheet. Then starting, for instance, at the bottom sheet and making a complete circle around a branch point (not encircling any other) we arrive at the top sheet (Figure 1).
The exact position of the branch cut depends essentially on the criterion used to distinguish the two possible polarizations.

A generally adopted way of separating both Riemann–sheets is to use the two signs of the root in Equation (1.3), thus

$$\left.\begin{aligned} Q_+ &= \frac{i\zeta}{1+\sqrt{1+\zeta^2}} = \frac{1-\sqrt{1+\zeta^2}}{i\zeta} \\ Q_- &= \frac{i\zeta}{1-\sqrt{1+\zeta^2}} = \frac{1+\sqrt{1+\zeta^2}}{i\zeta} \end{aligned}\right\} \tag{2.1}$$

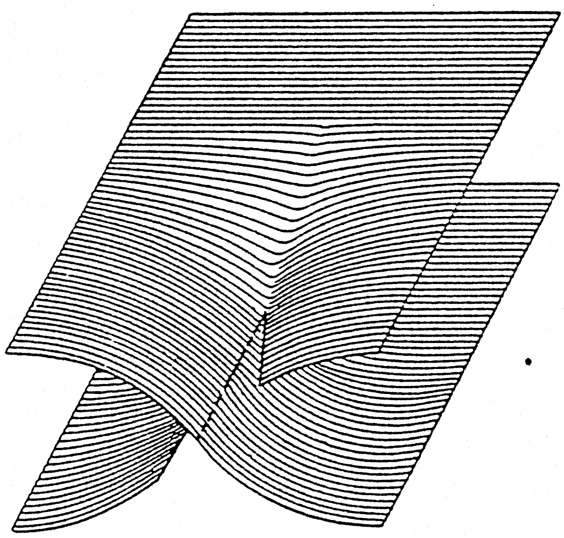


Figure 1: A double sheet Riemann surface. With the aid of conformal mapping onto a double sheet Riemann surface ionic and electronic polarizations can be completely separated. Electronic and ionic sheet touch each other at the so–called coupling points (branch points). At these points both characteristic polarizations are identical. The branch cut runs from $-i$ through zero to $+i$. The whole transformation can be split up into a rotation, an inversion at the unit circle and the subtransformation $w = \frac{1}{2}(z + 1/z)$. The latter transforms a grid of circles and lines into a grid of confocal ellipses and hyperbola [Bierbach, 1967].

Traditionally, Q_+ is called the polarization of the so–called 'ordinary' wave whereas Q_- is termed the polarization of the so–called 'extraordinary' wave.
From Equation (2.1) we immediately arrive at

$$Q_+ Q_- = 1\,. \tag{2.2}$$

This results in the entire realm of ordinary polarization lying inside the unit circle. If this distinction were used, we would be interested in a branch cut corresponding to the unit circle in the complex Q–plane (Figure 2). From a physical point of view, however, it would be much more satisfactory to distinguish the sense of rotation.
In doing so, one can distinguish ionic from electronic waves, i.e. ionic from electronic polarization. In that case the forced rotation of the charges with their natural gyration are compared. Using this definition to separate the two Riemann sheets, we will now try to obtain a branch cut corresponding to the real Q–axis of the complex Q–plane (see Figure 2). In view of what has just been said it might be most instructive to analyse as to how a grid consisting of circles $\mid Q \mid = \rho$ around the origin and lines $\arg\{Q\} = \varphi$ through the origin in the complex Q–plane is mapped onto the Riemann ζ–surface.

3 Subtransformations

For the sake of transparency, the whole mapping will be split up into three 'subtransformations'

1. $w = \frac{1}{2}(Q + 1/Q) \qquad \text{with } w \equiv \frac{1}{i\zeta}\,,$
2. $\tilde{w} = i\{\frac{1}{2}(Q + 1/Q)\} = iw = e^{i\frac{\pi}{2}}w \qquad \text{with } \tilde{w} \equiv \frac{1}{\zeta}\,,$

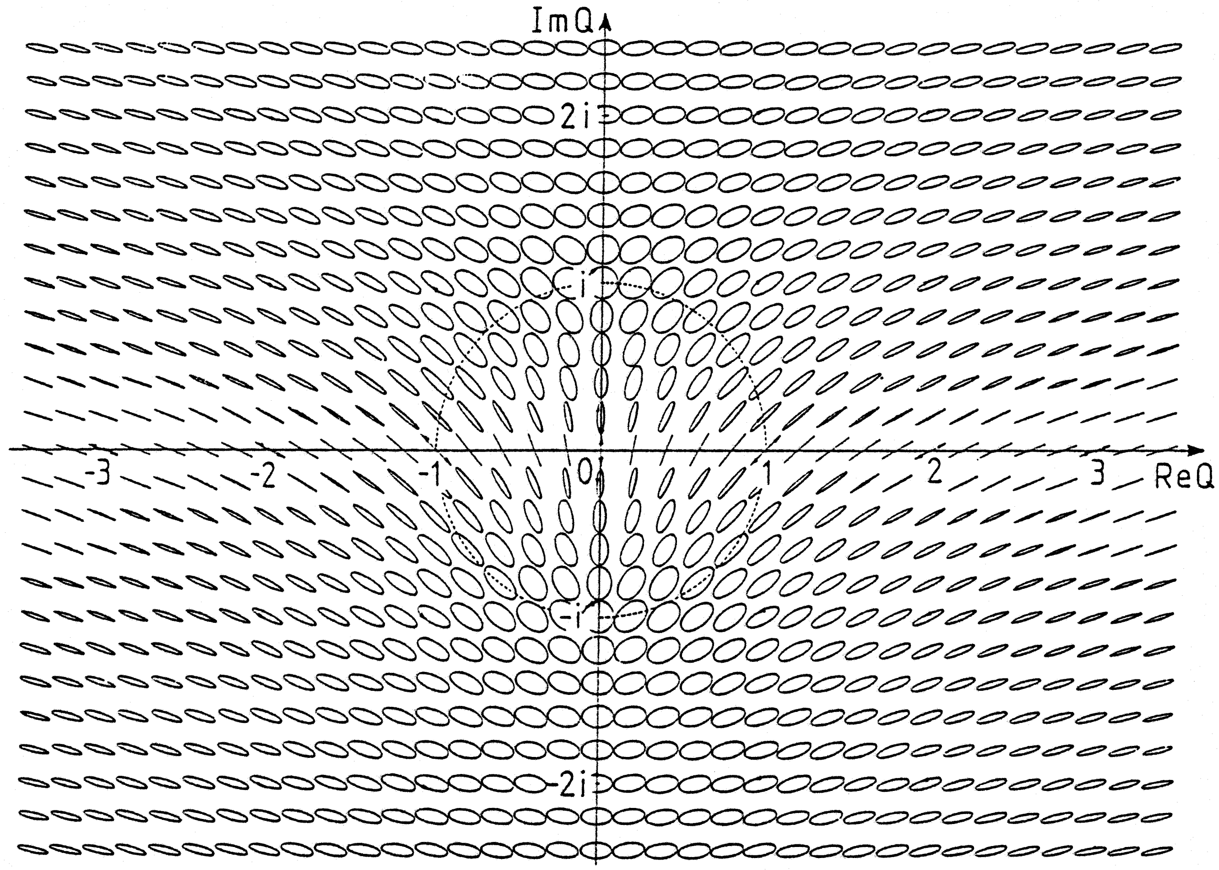


Figure 2: Booker's complex plane for $Q = E_x/E_y$ with polarization ellipses [Rawer and Suchy, 1967]. It can be seen that the sense of rotation is different in the lower and upper half plane. Comparing the sense of rotation with the natural gyration of electrons and positive ions, the upper half plane corresponds to ionic polarization whereas the lower half plane corresponds to electronic polarization. It should be emphasized that the magnetic field is supposed to be parallel to the index vector, i.e. only the longitudinal component of the Earth's magnetic field vector is considered. Furthermore, the unit circle is plotted distinguishing ordinary from extraordinary polarization.

3. $\zeta = 1/\tilde{w}$.

ad 1)

By introducing polar coordinates ($Q = \rho e^{i\varphi}$) and separating the real and the imaginary parts of w, one gets

$$\left.\begin{aligned} u &= (\rho + 1/\rho)\cos\varphi \\ v &= (\rho - 1/\rho)\sin\varphi \end{aligned}\right\} \tag{3.1}$$

where $w = u + iv$.
Equation (3.1) is nothing but a parametric equation of an ellipse. Hence a circle (ρ_0) in the Q–plane is transformed into an ellipse with semiaxes

$$a = \rho_0 + \frac{1}{\rho_0}\,, \qquad b = \mid \rho_0 - \frac{1}{\rho_0} \mid \ . \tag{3.2}$$

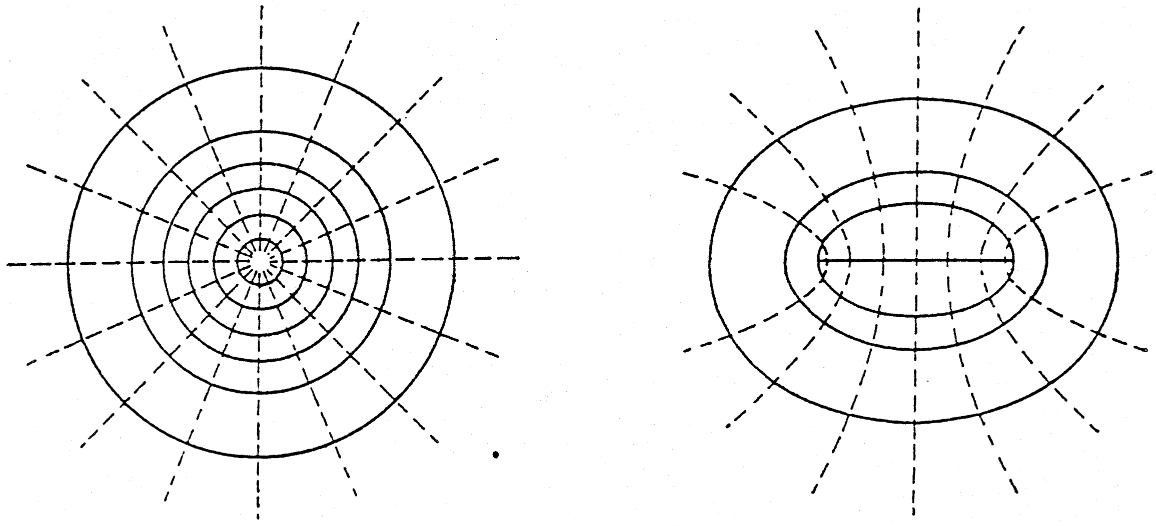


Figure 3: The subtransformation $w = \frac{1}{2}(z + 1/z)$ [Smirnow, 1987].

It is quite obvious that the circles with radii $\rho \neq 1$ and $1/\rho$ are mapped into the same ellipse. This is proved by the fact that the entire realm inside the unit circle corresponds to one Riemann sheet, whereas the entire realm outside the unit circle corresponds to the other Riemann sheet. The unit circle itself is transformed into the straight line $(-1, +1)$ on the real w–axis (see Figure 3). It is worthwhile noting that all the ellipses are confocal with the foci at the branch points $w = \pm 1$.

On the other hand, lines $\arg\{Q\}$ =const are mapped into a set of confocal hyperbola with the foci at $w = \pm 1$. The hyperbola corresponding to lines $\arg\{Q\} = 0$ and $\arg\{Q\} = \pi$ degenerate into intervals $(-\infty, -1)$ and $(+1, +\infty)$ on the real w–axis. A graphical representation of the transformation $w = \frac{1}{2}(Q + 1/Q)$ is given in Figure 3.

ad 2)

If the transformation $\tilde{w} = iw$ is taken into account, the grid of confocal ellipses and hyperbola is rotated by 90 degrees so that the Riemann surface of ζ has two branch points, viz. $w = \pm i$.

ad 3)

The last step $(1/\zeta)$ is equivalent to an inversion at the unit circle.

Summing up, it may be said that the unit circle in the complex Q plane is transformed into a line on the ζ–plane which runs from $+i$ along the imaginary axis through $\pm i\infty$ to $-i$ (inversion!).This would be the appropriate branch cut if ordinary (Q_+) and extraordinary (Q_-) polarization were to be separated. As mentioned above, we would like to distinguish Q_{ionic} from $Q_{electronic}$. It is, therefore, necessary to choose the representation of the real Q axis as a branch cut on the ζ–surface. This is the line from $+i$ through zero to $-i$. Both sheets are represented graphically in Figure 4.

So far purely mathematical considerations concerning multivalued functions have been taken into account.

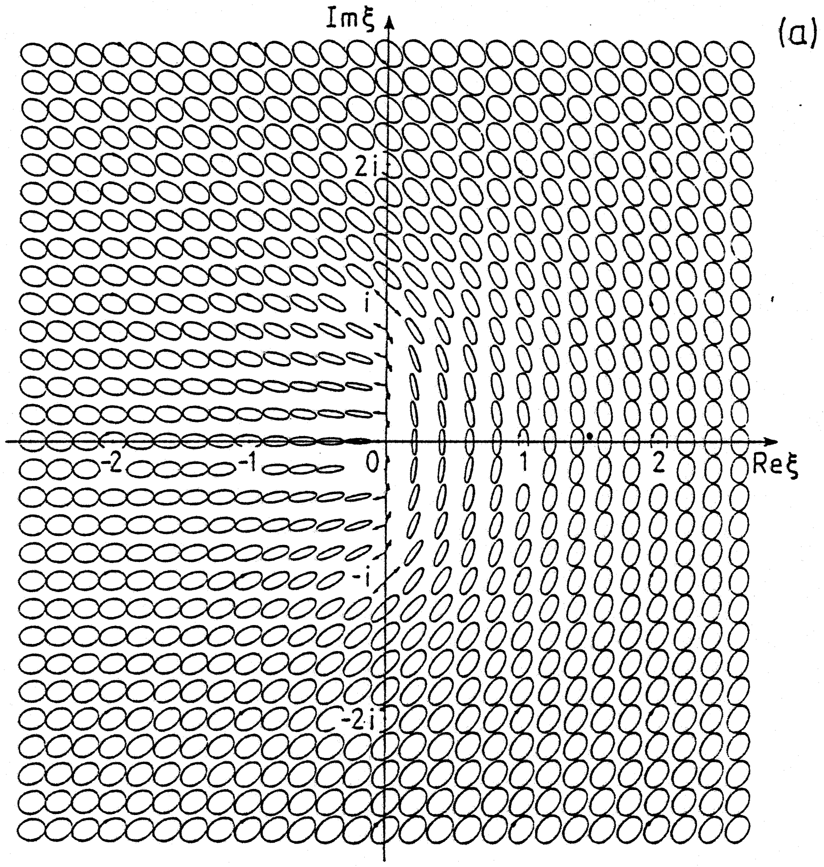
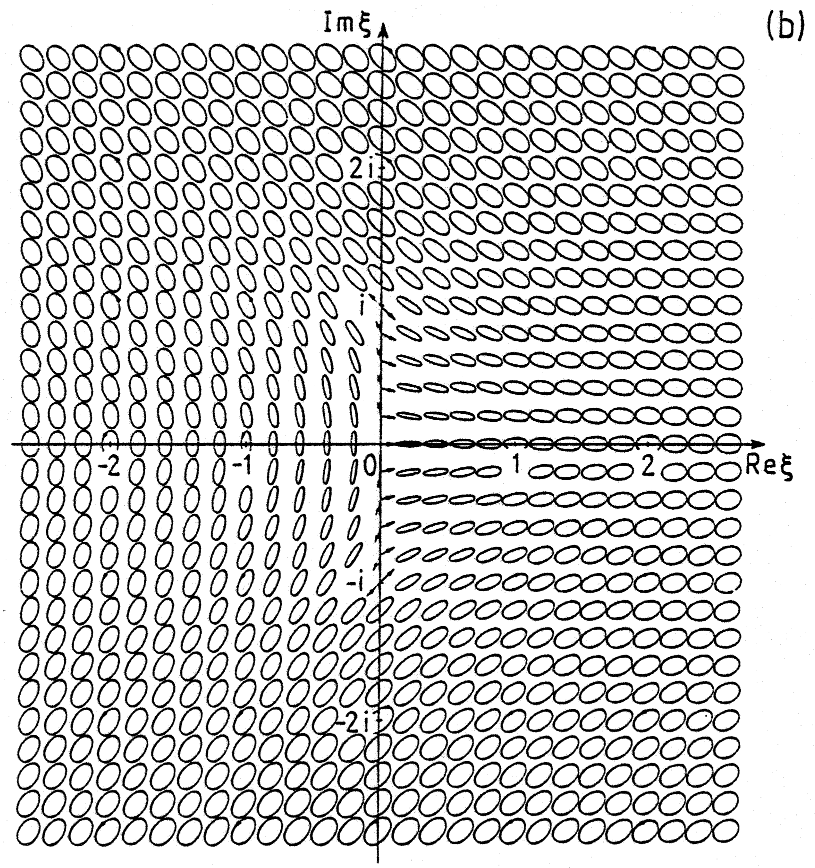


Figure 4: (a) Riemann surface for ζ with polarization ellipses. Sheet for ionic polariztion, $\vec{B}_{\oplus}$ emerges from the drawing plane.



(b) Sheet for electronic polarization [Rawer and Suchy, 1967].

4 Wave Path on Riemann ζ–Surface

We should now try to study the wave path on the Riemann ζ–surface. Compared with other forms of representation this has a major advantage as every change of polarization along the whole wave path can be followed.
It is important to note that, as soon as collisions are allowed, the eigenvalues of the dielectric tensor are complex so that parameter ζ has a finite imaginary part

$$\zeta = \Re\{\zeta\} + i\Im\{\zeta\}\,. \tag{4.1}$$

Furthermore, the dielectric eigenvalues change continuously in a slowly varying medium resulting in

$$\zeta = \zeta(\vec{B}, N_e, \bar{\nu}, \omega, \vec{r})\,, \tag{4.2}$$

with magnetic induction $\vec{B}$, plasma density N_e, collision frequency $\bar{\nu}$, wave frequency ω, and position $\vec{r}$.

In this way the 'equivalent' wave path on the ζ–surface for a known medium can be computed. An example of an 'equivalent' wave path for a cold electron plasma, where the collision frequency is assumed to be constant is shown in Figure 5. If this 'equivalent' wave path reaches a branch point, the condition for exact 'crossover' is fulfilled

$$\Re\{\zeta\} = 0 \qquad \text{and} \qquad \Im\{\zeta\} = \pm 1\,. \tag{4.3}$$

In case of Equation (4.3) both characteristic polarizations are equivalent. Physically, this means strong coupling between the two eigenmodes and, as a matter of fact, the branch point is very often called 'coupling point' or 'mode conversion point'.
Remembering that $\Im\{\zeta\} \propto \bar{\nu}$, the condition of exact 'crossover', can be given as

$$\Re\{\zeta\} = 0 \qquad \text{and} \qquad \bar{\nu} = \nu_c\,, \tag{4.4}$$

where ν_c is the critical collision frequency.

The coupling point can also be reached when

$$\Re\{\zeta\} = 0 \qquad \text{and} \qquad \theta = \theta_c\,, \tag{4.5}$$

where

$$\frac{\sin^2\theta_c}{\cos\theta_c} = \pm\Im\{\zeta\}\,.$$

For $\mid \Im\{\zeta\} \mid > 1$, i.e. $\bar{\nu} > \nu_c$ (equivalent to $\theta < \theta_c$) the wave crosses the imaginary axis above the branch point. Thus no transition occurs. The adjectives used to describe the wave are not changed (ionic/ electronic). If we had adopted a plus/minus definition of polarization, a transition would occur just in this case.

If collisions are negligible, the eigenvalues of the dielectric tensor are real and the parameter ζ has no imaginary part. Nevertheless—under certain circumstances—coupling effects

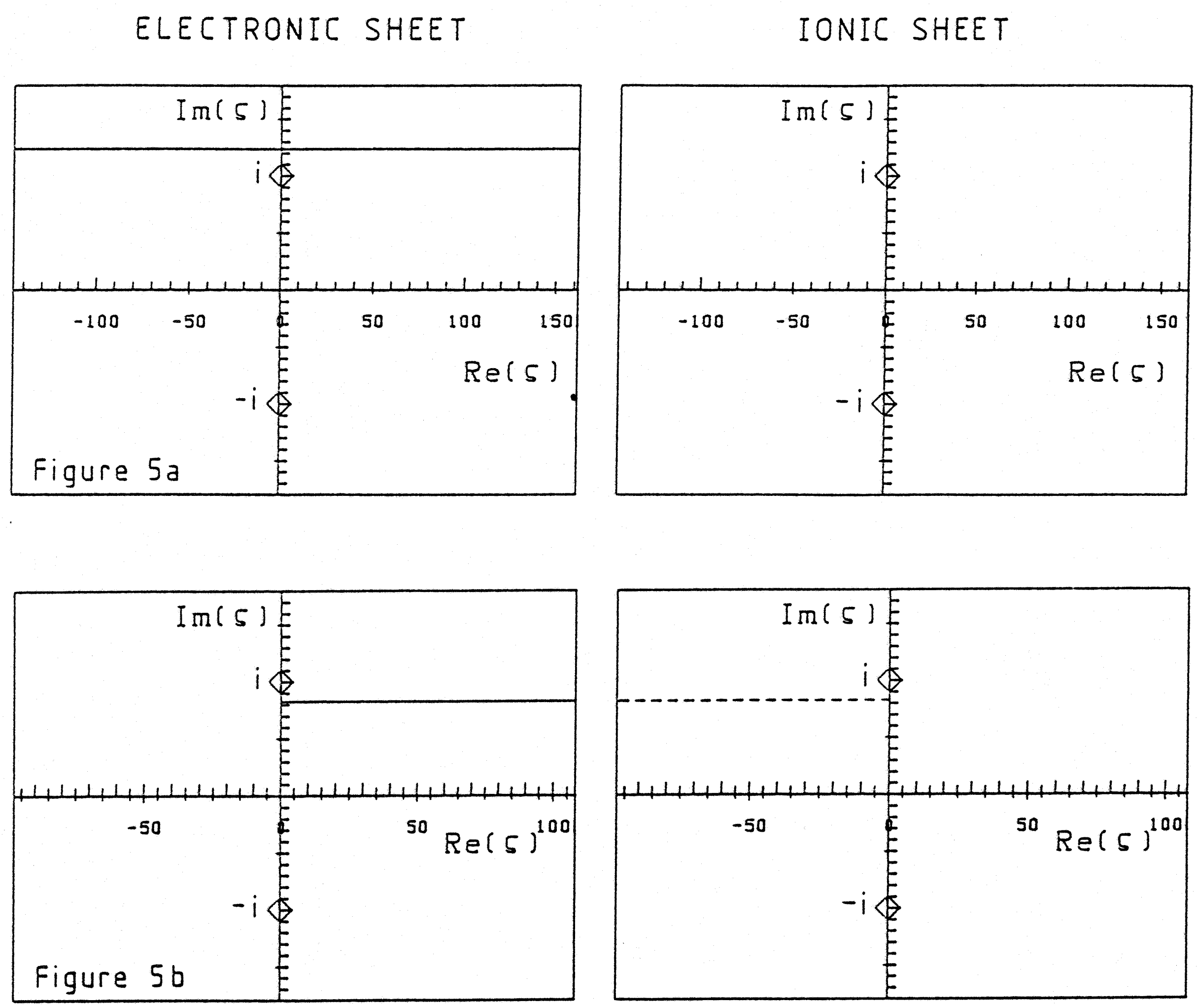


Figure 5: (a) ◇ ... branch points (coupling points), $\theta = 9°$ (θ is the angle between magnetic field vector and index vector);
(b) ◇ ... branch points, $\theta = 11°$. The figures show the 'equivalent' wave path on a double sheet Riemann ζ–surface for an ascending electronic wave in the southern hemisphere. The parameters describing the slowly varying medium (cold electron plasma) are chosen such that transition occurs at $\theta = \theta_c = 10°$. Figure 5a shows the wave path for $\theta = 9°$ which is slightly less than θ_c. As a matter of fact the curve crosses the imaginary axis above the branch cut and no change of labeling is necessary, i.e. the wave remains an electronic wave. Figure 5b, on the contrary, shows the wave path for $\theta = 11°$ which is slightly greater than θ_c. It can be seen that the curve crosses the branch cut. Mathematically this means a transition from the electronic to the ionic sheet and therefore the description of the wave has to be changed.

are possible. In order to examine these, we seek non–trivial solutions of the equation $1 + \zeta^2 = 0$ (see Equation (1.3)). Inserting the parameter ζ , one gets

$$(RL - PS)^2 \sin^4\theta + 4P^2D^2\cos^2\theta = 0\,. \tag{4.6}$$

This equation holds if

$$\theta = 0 \qquad \text{and} \qquad PD = 0\,, \tag{4.7}$$

$$\theta = \frac{\pi}{2} \qquad \text{and} \qquad RL = PS\,. \tag{4.8}$$

If these conditions are fulfilled exact crossover occurs in a cold collisionless plasma and the labeling of the wave has to be changed. It should be said that $D = 0$ does not occur in a plasma containing only one species of ions, except if conditions are $\vec{B} = 0$ or $N_e = 0$. Figure 4 shows that $\theta = 0$, i.e. $\zeta = \infty$, corresponds to circularily polarized waves whereas $\theta = \frac{\pi}{2}$, i.e. $\zeta = 0$, corresponds to linearly polarized waves.

5 Conclusion

1. The method of conformal mapping is most helpful especially when operating with double valued functions such as polarization or refractive index.
2. Based on a model ionosphere the wave path on the Riemann ζ–surface can be computed. With the help of such a representation any change of polarization, that is shape and orientation of the polarization ellipse, and the sense of rotation of the field vector, along the whole wave path can be studied.
3. It is possible to extend the whole theory so that effects such as cutoffs and resonances can be included.

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References

Bierbach, L., Einführung in die konforme Abbildung, Sammlung *Göschen Band*, 768/768a, Walter de Gruyter & Co., Berlin, 1967.

Rawer, K., K., Suchy, Radio–Observations of the Ionosphere, *Handbuch der Physik*, XLIX/2, Geophysik III/2, Springer Verlag, 1967.

Smirnow, W. I., Lehrgang der höheren Mathematik, Bd. 4, Teil III/2, VEB Deutscher Verlag der Wissenschaften, Berlin, 1987.

Stix, T. H., The Theory of Plasma Waves, McGraw–Hill, New York, 1962.